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Efficiency of composite fins of variable thickness

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Abstract

This paper discusses the thermal calculation of composite, metallic fins of variable thickness. In the simpler case of a constant-thickness (rectangular profile), the complete procedure involves first the analytical solution of the two-dimensional, two-material conduction problem, under the form of an infinite series of orthogonal eigenfunctions. Then the limit as $Bi \to 0$ is sought, also analytically, which simplifies the series to its first term and permits to express the fin efficiency in closed form. This limit is equivalent to the usual 1D, Murray–Gardner, or thin-fin, approximation of ordinary, single-material fins, provided that an averaged thermal conductivity is used.

For variable thickness (tapered profile), no analytical solutions have been found, so that resort should be made to numerical methods. Since the adoption of dimensional parameters is advisable in that context, the paper first reviews the industrial application of composite fins and selects a comprehensive set of material pairs of interest. Subsequently, two arbitrary but representative geometries and a reasonable range of dimensions and convection coefficients are fixed, thus assembling a rather exhaustive matrix of case-studies. Numerical calculations are compared to approximate results, in order to ascertain two facts: whether a single parameter exists (thermal length) that allows an accurate prediction of fin efficiency, and whether this parameter can be expressed in terms of an averaged thermal conductivity. Well within the bounds of usual engineering accuracy, the answer to both questions is affirmative. Therefore, calculation methods of ordinary fins and composite, constant-thickness fins are shown to be applicable to the most general case. Error bounds and specific recommendations for practical problems are also given.

Keywords: Fin; Composite fins; Fin efficiency; Heat transfer augmentation

1. Introduction

The concept of a composite fin, made of a substrate and a coating of two different materials, has received a scarce, almost perfunctory, treatment in the heat transfer literature. The subject appeared very early indeed, with the studies of Barker [1,2] on the problem of constant-thickness in Cartesian and axial-cylindrical coordinates, i.e., straight and pin-type fins. Barker used the theory of orthogonal expansions [3] and found out the exact, 2D series solution. By analyzing the eigenvalues, he was subsequently able to show that the series converged to its first term for a Biot number sufficiently small, and that this approximation was equivalent to the usual 1D treatment if a cross-sectional area average of the thermal conductivities is used. In other words, approximate 1D conduction still prevails under the usual slender-fin conditions, even in the case of a composite of two different materials.

Only a few studies were published afterwards. The exact solution for the annular fin of constant-thickness was given in [4], and compared to the approximation of Barker adapted to that geometry. It was concluded that the 2D calculation was necessary in many instances, but the range of situations included Biot numbers and ratios of thicknesses much larger than it is normal in metallic, dissipative fin applications. More recently, Lalot et al. [5] rediscovered

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Nomenclature

A	cross-sectional area, m^2
Bi	Biot number, ht/k
$f(\ldots)$	unspecified function
h	convection coefficient, W/m^2 K
k	thermal conductivity, W/m K
L	fin length, m
m	fin parameter, m^{-1}
mL	fin thermal length
q	heat transfer rate, W
q'	heat transfer rate per unit length, W/m
r	radius, m
T	temperature, °C
t	fin half-thickness or coating thickness, m
V	volume, m^3
x	longitudinal coordinate, m
y	fin half-thickness function, m

Greek symbols

ε	error (%)
η	fin efficiency
λ_i	eigenvalue, m^{-1}
θ	dimensionless temperature, $(T - T_\infty)/(T_\mathrm{b} - T_\infty)$

Subscripts

a	analytical
av	averaged
b	fin base
c	coating
eq	equivalent, as defined in relation to Eq. (21)
n	numerical
s	substrate
t	fin tip
∞	external fluid

Superscript

*	dimensionless

the same problem. In a rather contorted, but interesting way, that avoided the derivation of an infinite series, they were able to parallel the eigenvalue analysis of Barker. The result was analogous: the fin can be treated as an ordinary, single-material fin by the intuitive expedient of averaging thermal conductivities in proportion to cross-sectional areas.[1]

Lalot et al. also quantified the accuracy of the approximation and found it excellent (0.01% in fin efficiency) for the usual range of parameters found in metallic fins (thicknesses up to 1 mm; maximum thickness ratio around 50%; conductivity ratios near unity; Bi of the order of thousandths). Other studies have also found a good accuracy in similar applications [6,7]. In [6], the range of conductivity ratios was extended up to 20 by considering different material pairs. Campo [7] additionally postulated, on statistical grounds, that the arithmetic average of thermal conductivities is always an upper bound for the actual fin performance.

All these studies dealt with a composite fin made of two metals, for heat transfer enhancement or protective purposes. There is also the theoretically related problem of an ordinary fin performing under fouled or frosted conditions, in which the coating is a thermally insulating material and higher thickness ratios may exist. Although the 2D solution is the same, its reduction to an engineering approximation is completely different, due to the fact that the diverging values of conductivities and thicknesses make ambiguous the condition $Bi \to 0$. Or in other words, it might be that an approximately 1D thermal field prevails in the substrate (fin) but not in the coating (insulation). For constant-thickness, an "intermediate" solution can then be sought, in which 1D and 2D temperatures are solved in the substrate and coating, respectively.[2] This problem was given a primitive treatment by means of "thermal resistances" in Refs. [8,9], the latter also advancing some numerical and analytical work. The strict mathematical theory was first developed in [10] for a straight fin and afterwards extended to the annular geometry in [11,12]. Sometimes, archival literature mixes this problem with that of a two-material fin, which can be confusing; a good example is Ref. [4] mentioned above.[3]

As for our problem of a composite, metallic fin, quantification of error bounds of the 1D approximation has not been rigorous nor exhaustive, perhaps due to the fact that an analytic limit was known that is explained very intuitively, and thus always presumed correct without questioning.[4] Other point of argument is the value of the conductivity ratio. Considering some applications (see

[1] The fact that they are variable is immaterial, since they are both proportional to the radius.

[2] Which necessitates a special application of the theory of orthogonal series in composite media, and eventually reduces itself to the usual approximation: a treatment of the insulation which is locally 1D in the transverse direction.

[3] Also in Ref. [13], a thin metallic cladding on a fin is given a 1D(coating)/2D(substrate) treatment, analytical and numerical, which is mostly of interest if the substrate is an insulation.

[4] Even fundamental studies are not devoid of ambiguities. For instance, the errors reported by Lalot et al. [5] refer to the difference in using exact or approximate figures for the first eigenvalue, which can be shown to be far less restrictive than to compare the full series, i.e., the exact solution, with its first term.

below), it can range over two or three orders-of-magnitude around unity, even for metals. To what extent is then applicable the 1D approximation in the terms imagined by Barker? Finally, two-material fins of variable thickness may be of interest for obvious reasons of volume optimization, as ordinary fins are. However, the analytic theory of composite media cannot handle this kind of problems. Therefore, there is not a limiting process that could extract a 1D approximation from the general solution, and a clearly defined procedure of calculation is missing. Some studies have explored tentatively the rather evident expedient of generalizing the average conductivity and trying to qualify the approximation with reference to numerical solutions [6,14]. But, again, there is a lack of sound and systematic work in this respect.

In this paper, we attempt to give an answer to all these questions by means of the following sequence of considerations. Firstly, a review of fin theory from a general point of view is assembled, with the purpose of advancing a reasonable conjecture for the approximation of composite fins of variable thickness. Next, attention is directed towards industrial applications, reviewing specific materials used, which gives an ample interval of conductivity ratios. Since numerical calculations are intended, particular cases and typical ranges of dimensional parameters are needed in addition. We propose to analyze two families of geometries: the straight fin of lineal profile and the annular fin of hyperbolic profile, along with a reasonable range of geometric dimensions and heat transfer coefficients. This provides both specific cases to work with and practical ranges of dimensionless parameters (essentially the Biot number) within which the accuracy of the approximation is of interest.

Finally, results of the numerical, 2D calculations are compared with the 1D approximation and the ensuing errors are analyzed. Although based on a necessarily finite and arbitrary set of case-studies, results may serve to clarify whether simple calculations are accurate enough for a wide range of practical applications. In fact, the extreme values of conductivity ratios combined with a tapered longitudinal profile assure that any 1D approximation will incur the highest possible errors. Thus, the present study can be regarded also as a general assessment of the engineering treatment of composite fins of slender profile.

2. Fin theory and its application to composite fins

We will follow the simple and approximate assumptions of the usual analysis of fins. These are: (a) steady state; (b) no internal heat sources; (c) isotropic and homogeneous medium and constant thermal conductivity k; (d) uniform and linear boundary conditions – constant values of temperature of the fin base T_b, surrounding fluid temperature T_∞ and convection coefficient h; (e) negligible heat transfer from the fin tip; (f) symmetrically adiabatic mid-plane. In addition, for composite fins: (g) nil contact resistance between dissimilar materials. We will assume all these valid either for a 2D or an approximate 1D problem. An adiabatic tip seems to imply some limitation, since it is not generally warranted in 2D. However, the generalization would only necessitate to show that tip heat transfer becomes negligible along with cross-sectional area for a slender geometry, a side question which is best avoided.

The basic theory of ordinary fins can be discussed in the following terms. Consider as an example a straight fin of half-thickness t and length L. If all parameters are accounted for, the heat dissipation per unit width q' (W/m) must be

$$q' = f(t, L, k, h, T_b - T_\infty) \tag{1}$$

where we use the notation $f(\ldots)$ to signify some generic function of the specified arguments. It should be noted that the problem is linear and homogeneous by virtue of hypotheses (b), (c) and (d) above, so that only the difference $T_b - T_\infty$ appears as a parameter. By simple dimensional reasoning, Eq. (1) can be made dimensionless in this way:

$$\eta = \frac{q'}{hL(T_b - T_\infty)} = f\left(\frac{ht}{k}, \frac{t}{L}\right) = f\left(Bi, \frac{t}{L}\right) \tag{2}$$

where η is the so-called fin efficiency, usually interpreted as the fraction of maximum possible heat transfer attained. Fin performance thus depends only on two parameters: the Biot number Bi and the geometric ratio t/L. However, as it is commonly known, this relation is further simplified under the 1D approximation, to give

$$\eta = f(mL) \tag{3}$$

where the quantity mL is dimensionless and m (m^{-1}) is defined as

$$m = \left(\frac{h}{kt}\right)^{\frac{1}{2}} \tag{4}$$

Frequently, mL is conceived as the "thermal length" of the fin. Other interpretation consists in realizing that it is simply a particular combination of the original dimensionless parameters:

$$mL = \left(\frac{h}{kt}\right)^{\frac{1}{2}} L = \left[\frac{ht}{k}\left(\frac{L}{t}\right)^2\right]^{\frac{1}{2}} = \frac{Bi^{\frac{1}{2}}}{t/L} \tag{5}$$

The reduction of Eq. (2) to Eq. (3) can be given diverse explanations. For the case of constant-thickness, a complete analytic rendition exists, as put forward by Levitsky [15]. Since the fin has a finite thickness t, the exact temperature field is two-dimensional and, if t is a constant, the problem is solvable by the method of separation of variables. The exact formula for the fin efficiency turns out to be

$$\eta = \sum_{i=1}^{\infty} C_i \frac{\tanh(\lambda_i L)}{\lambda_i L} \cos(\lambda_i t) \tag{6}$$

In this expression, λ_i represents an infinite number of positive eigenvalues given by the implicit relation $\lambda_i t \tan(\lambda_i t) = Bi$, i.e., in such a way that the series $\lambda_i t$ is only a function of Bi. Also, the series of constants C_i is an exclusive function of $\lambda_i t$. Therefore, the formula complies with the format of Eq. (2). On the other hand, if one assumes a 1D temperature field, a simple, ordinary differential equation can be written and solved, from which fin efficiency is derived in a form compliant with Eq. (3):

$$\eta = \frac{\tanh(mL)}{mL} \tag{7}$$

The simplification of Eq. (6) to Eq. (7) can be derived as an analytical limit.[5] In an order-of-magnitude sense, the Biot number $Bi = ht/k$ modulates the local, transverse temperature differences inside and outside the material, so that when $Bi \to 0$, the fin is locally isothermal. Therefore, the 1D approximation amounts to assume a vanishingly small Biot number. Under this circumstance, it follows that $\lambda_i t \to Bi^{1/2} = mt \to 0$ by virtue of the eigenvalue relation, and that $C_1 \to 1$, $C_i \to 0$ for $i > 1$. Substituting in Eq. (6), the infinite series of the exact solution reduces to its first term, and this coincides in turn with Eq. (7). It should be noted in passing that the analysis of Levitsky was originally aimed at proving that the condition for a slender, 1D-tractable fin is $Bi \to 0$, and not simply the geometric condition $t/L \to 0$. However, for normal metallic applications, thermally slender fins are also geometrically slender, or, according to Eq. (5), their thermal length is not small.

Eqs. (1)–(5) and their basic meaning continue to hold for a straight fin of variable thickness, whose half-profile follows some given function $y(x)$. For specific $y(x)$, advanced analytical methods allow to obtain 2D solutions. However, their form is such that an expression analogous to Eq. (6) cannot be written, which dismisses the existence of a series solution and thus of an analytical, easily derived, limit. But if we accept an approximate 1D flow of heat, the equation for the temperature field $\theta(x)$ is [17]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) - \frac{h}{k}\theta = 0 \tag{8}$$

Solutions $\theta(x)$ are available in the literature for specific functions $y(x)$. For a generic $y(x)$, the basic dependency can be demonstrated as follows. First, redefine the parameter t as the half-thickness at a fixed x, for instance the root of the fin, and use L and t to make x and $y(x)$ dimensionless, respectively: $x^* = x/L$, $y^* = y/t$. Eq. (8) is then rearranged as

$$\frac{\mathrm{d}}{\mathrm{d}x^*}\left(y^*\frac{\mathrm{d}\theta}{\mathrm{d}x^*}\right) - \frac{h}{kt}L^2\theta = 0 \tag{9}$$

Adopting the definition of Eq. (4), the last coefficient is m^2L^2. Therefore, since boundary conditions do not introduce any additional parameter, the temperature θ is only a function of x^* and mL, for a given dimensionless profile $y^*(x^*)$. The fin efficiency is now written as

$$\eta = \frac{q'}{hL(T_b - T_\infty)} = \frac{\int_0^L h(T - T_\infty)\,\mathrm{d}x}{hL(T_b - T_\infty)} = \int_0^1 \theta\,\mathrm{d}x^* \tag{10}$$

and, since $\theta = f(x^*, mL)$, it follows that $\eta = f(mL)$, conforming to Eq. (3). Of course, η will depend additionally on any geometric variable introduced by the dimensionless function $y^*(x^*)$. For instance, for realistic fins with finite tip thickness, on the ratio t_t/t_b.

There is however a condition that may be regarded as supplementary to $Bi \to 0$. Eqs. (8) and (9) substitute the length-of-arc of the profile $y(x)$ by x and its total length by L, which is usually called the "length-of-arc assumption" [18]. For a geometrically slender-fin, this cannot be considered as an independent restriction, obviously, but it can be for a thermally slender one, and then η still depends on t/L. In any case, as we have seen, this is not important for practical fins, so that we will adopt the assumption in what follows.

Although presented for straight fins, the discussion would have been similar for the annular and spine geometries. The radii ratio r_t/r_b is an additional parameter in the first case, obviously, and Eqs. (6)–(10) would have had different forms, but the ideas of the 1D approximation and the ensuing thermal length, Eqs. (3) and (4), are exactly the same.

For a composite of two dissimilar materials, an analogous discussion can be pursued, at least partially. Listing parameters, the exact efficiency will be for instance, for the straight fin of constant-thickness:

$$\eta = f\left(Bi, \frac{t_s + t_c}{L}, \frac{t_c}{t_s}, \frac{k_c}{k_s}\right) \tag{11}$$

where the subscript s denotes the fin substrate and the subscript c denotes the coating. Under the 1D approximation, one may speculate that Eq. (11) reduces to one of these equations

$$\eta \approx f\left(m_{av}L, \frac{t_c}{t_s}, \frac{k_c}{k_s}\right) \tag{12a}$$

$$\eta \approx f(m_{av}L) \tag{12b}$$

$$\eta \approx \frac{\tanh(m_{av}L)}{m_{av}L} \tag{12c}$$

with

$$m_{av} = \left[\frac{h}{k_{av}(t_s + t_c)}\right]^{\frac{1}{2}} \tag{13}$$

In other words, perhaps an effective or averaged thermal length may be defined with the total half-thickness $t_s + t_c$ and some sort of average k_{av} of the thermal conductivities of the two materials. With that average, it might be that (a) the overall thermal and geometrical parameters combine but η be still a function of the material thermal and geometric ratios, (b) additionally η be independent of those material ratios, or (c) additionally η be *the same function of the thermal length as the ordinary fin of the same geometry*.

[5] In fact, a limit analogous to that found in the reduction of the Fourier problem to a lumped capacitance transient, see for instance [16].

As before, the case of constant-thickness has an analytical 2D solution obtainable by the method of orthogonal expansions in composite media [2], i.e., a generalization of Eq. (6) exists that gives the exact efficiency of the composite fin according to the format of Eq. (11). Similarly, its simplification as $Bi \to 0$ is a more involved process, but analogous to that described above. It elegantly comes out with the simpler option of Eq. (12c), in favor of the more complicated Eqs. (12a) and (12b). With regard to the exact form of the parameter k_{av}, it is an arithmetic average weighted with the thicknesses of the materials for the straight [2] and annular [5] fins, and with the square radius and difference of square radii for the pin fin [2]. Therefore, it can be interpreted as an average in proportion to the cross-sectional areas:

$$k_{av} = \frac{k_s A_s + k_c A_c}{A_s + A_c} \tag{14}$$

For composite fins of variable thickness, there are not analytical 2D solutions available, but if approximate 1D conduction is accepted, substrate (s) and coating (c) temperatures are forcibly equal, obeying the equation

$$\frac{d}{dx}\left\{[k_s y_s + k_c(y_c - y_s)]\left(\frac{d\theta}{dx}\right)\right\} - h\theta = 0 \tag{15}$$

Now analytical results are understandable from a simpler perspective: for constant-thickness, y_s, $y_c = \text{const.}$, the factor multiplying $d\theta/dx$ in Eq. (15) leaves the derivative sign and we recover Eq. (8) with an apparent conductivity given by Eq. (14) – and also a constant $y = y_c$.

For variable thickness and simple forms of the functions $y_s(x)$, $y_c(x)$, Eq. (15) will surely admit analytical solutions dependent on the thermal and geometric ratios k_c/k_s, t_c/t_s. But there is a prospect more interesting to explore. Is it possible that, for *any* form of the profile, effective or averaged fin thermal conductivity and length exist that render the calculation as intuitive as in the case of constant-thickness?

As laid out in the introduction, we investigate numerically the question, trying to ascertain if Eqs. (13) and (14) may serve in the general case to accurately calculate fin efficiency, and to what level of simplicity, among those exemplified by Eq. (12). As a reasonable hypothesis, and based on preliminary studies, we generalize Eq. (14) by the simple expedient of taking average cross-sectional areas, which is equivalent to a volume–weighted average:

$$k_{av} = \frac{k_s V_s + k_c V_c}{V_s + V_c} \tag{16}$$

We attempt to ascertain however if better alternatives exist. Geometries tested are selected rather arbitrarily, but dimensions and thermal parameters are representative of industrial practice. In relation with this, there is an important point. The thermal ratio k_c/k_s of practical, composite, metallic fins can differ widely from unity. This may render confuse the $Bi \to 0$ limit necessary for enforcing the 1D approximation because Bi can be indeed small for one material but large for the other. Order-of-magnitude reasoning may then become awkward – see Ref. [10] for a good example. Since our study is based on numerical, 2D calculations that accurately approximate the exact solution, we take the path of least resistance in this respect. Extreme (though reasonable) values of k_c/k_s are adopted, and the study is simply extended to them, to see if the composite can still be calculated by the 1D fin approximation. Moreover, we can also speculate if an average definition of Bi could suffice itself to establish how closely the approximation is met. In other words, we will put under test a Biot number defined as

$$Bi_{av} = \frac{h(t_{s,b} + t_c)}{k_{av}} \tag{17}$$

where a reasonable (though arbitrary) characteristic length based on the total thickness at the fin base is used.

3. Composite fins: applications and materials

We may perfectly classify the applications by the magnitude of the thermal conductivity ratio k_c/k_s. If it is greater than unity, then the coating material is more conductive than the substrate. A good example is found in the conventional industrial techniques of galvanizing or aluminizing, where a steel piece is coated by a thin layer of zinc or aluminum. To impregnate this layer, the final phase of the manufacturing process requires the immersion of the substrate in a bath of liquid coating [19]. The usual goal is to provide a carbon steel frame with a self-protective layer against corrosive environments. Application of this kind of composite fins is normally seen in high-performance exchangers, in which an enhanced heat transfer is obtained as a side-effect [5]. This kind of material combinations has been selected for the simulations, yielding conductivity ratios moderately greater than one. To increase the range of k_c/k_s, also an aluminum coating on a stainless steel substrate has been considered. The thermal spray technique makes possible the deposition of aluminum multi-microlayers on this surface material, as well as on carbon steel and aluminum alloys [20].

The opposite situation of a coating of lower thermal conductivity than the substrate is largely found in finned, compact heat exchangers framed with a copper alloy but covered with a thin, protective layer of stainless steel. The copper is selected due to its high thermal conductivity and ease of fabrication; however, it is weak and oxidizes rapidly at high temperatures. The copper is protected and strengthened by a coating of stainless steel, which can be metallurgically bonded to the substrate by electroplating or dipping [20,21]. In the present work, two different substrates are analyzed, providing different conductivity ratios: a 15% Zn copper brass and a non-alloyed free-oxygen copper.

Table 1 summarizes the five combinations of coating/substrate materials selected, and the values considered for

Table 1
Material combinations and their thermal conductivities

Combination	Substrate	Coating	k_s (W/m K)	k_c (W/m K)	k_c/k_s
#1	Stainless steel, 20% Cr–15% Ni	Aluminum, 99.5%	15	204	13.60
#2	Carbon steel, 0.5% C	Aluminum, 99.5%	54	204	3.78
#3	Carbon steel, 0.5% C	Zinc, 99.9%	54	112	2.07
#4	Copper brass, 15% Zn	Stainless steel, 20% Cr–15% Ni	159	15	0.09
#5	Oxygen-free copper, 99.3%	Stainless steel, 20% Cr–15% Ni	376	15	0.04

their thermal conductivities [22]. Although this work is focused on the performance of metallic fins, the range of the ratio k_c/k_s is quite wide, from 0.04 to 13.6, perfectly compatible with other kinds of situations. For instance, a thin layer of ice on steel will be also tractable by the methods developed here.

4. Case-studies

Two geometries have been arbitrarily adopted as the starting point for defining different case-studies. Firstly, the basic linear variation in a two-dimensional geometry with realistic, finite tip is considered, i.e., a straight fin of trapezoidal profile. In the second place, the annular geometry and a non-linear profile are combined in the annular hyperbolic fin. Sketches are displayed in Figs. 1 and 2, representing the upper symmetric half and with a vertical scale greatly exaggerated.

A coating of constant-thickness t_c on a variable substrate has been assumed in any case, since it is the logical option for most manufacturing processes. Note that t_c is defined in the vertical direction, rather than normal to the surface. Given the geometrical slenderness of the fins, this is mostly indifferent, and has the advantage of a simpler coating geometry, of constant cross-sectional area in the case of a non-linear profile. Also for simplicity, the form of the profile is established for the total thickness y_c, and then the substrate form defined as $y_s = y_c - t_c$. This is of course not very compatible with an actual coating process on a non-linear profile, but again of little importance for thin geometries.

Dimensions of practical applications are, typically: substrate half-thickness $t_s = 0.1$–0.5 mm; coating thickness $t_c = 30$–80 μm; length $L = 5$–40 mm; radius at the base $r_b = 5$–15 mm, radii ratio $r_t/r_b = 1.5$–3. Starting from these reference figures, different dimensions have been simulated in the present work, according to the objectives of the investigation.

For the straight fin of trapezoidal profile, length has been prescribed equal to 10 mm and the following geometric parameters have been varied: substrate half-thickness at the base $t_{s,b} = 0.4$–0.6 mm; substrate half-thickness at the tip $t_{s,t} = 0.2$–0.3 mm; coating thickness $t_c = 40$–80 μm. In the case of the annular hyperbolic fin, length has also been fixed to 10 mm and the same ranges considered for $t_{s,b}$ and t_c, but fin aspect has been varied by changing the radius at the base r_b from 5 to 10 mm.

Tables 2 and 3 summarize the complete set of geometric arrangements and give the volume-averaged thermal conductivities for every material pair and case, calculated after Eq. (16). As the tables show, the substrate geometry remains the same for the first five cases, and the coating is progressively thickened. The following three cases in Table 3 correspond to a progressive reduction of the radius of the fin base, maintaining the same geometry for substrate and coating. In the two last cases of both tables, substrate and coating are proportionally thickened maintaining the same ratio t_c/t_s.

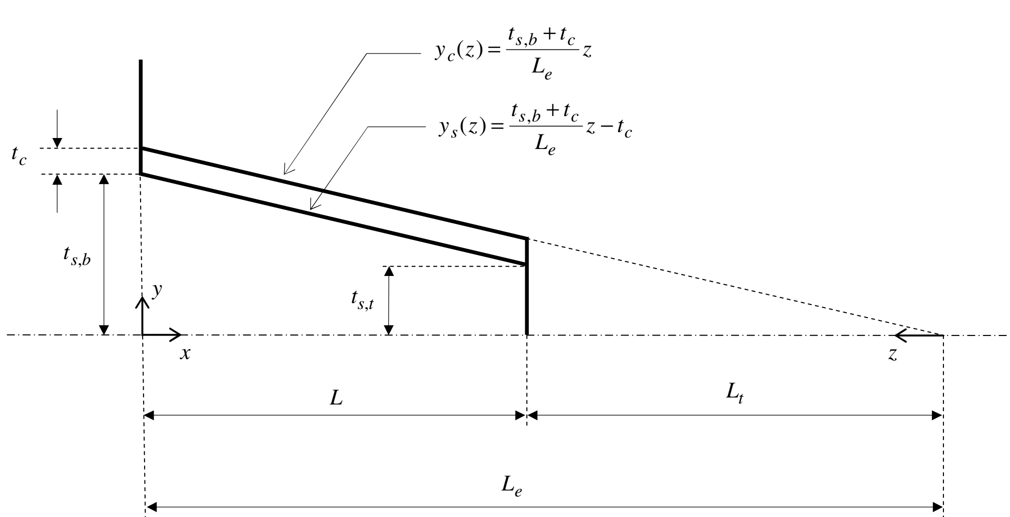

Fig. 1. Composite straight fin of trapezoidal profile.

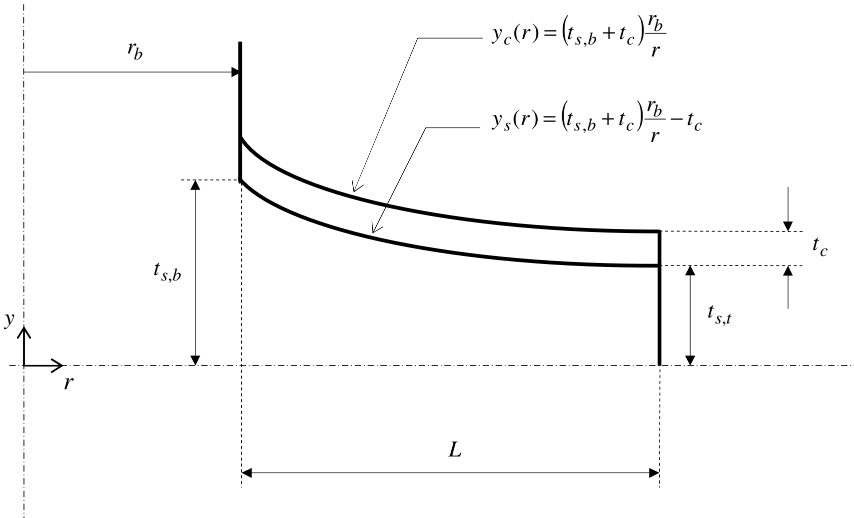

Fig. 2. Composite annular fin of hyperbolic profile.

Table 2
Set of geometric cases for the straight fin of trapezoidal profile ($L = 10$ mm)

$t_{s,b}$ (mm)	$t_{s,t}$ (mm)	t_c (mm)	k_{av} (W/m K)				
			#1	#2	#3	#4	#5
0.4	0.2	0.04	37.24	71.65	60.82	142.06	333.53
		0.05	42.00	75.43	62.29	138.43	324.43
		0.06	46.50	79.00	63.67	135.00	315.83
		0.07	50.76	82.38	64.97	131.76	307.70
		0.08	54.79	85.58	66.21	128.68	300.00
0.5	0.25	0.05	37.24	71.65	60.82	142.06	333.53
0.6	0.3	0.06	37.24	71.65	60.82	142.06	333.53

Table 3
Set of geometric cases for the annular fin of hyperbolic profile ($L = 10$ mm)

$t_{s,b}$ (mm)	t_c (mm)	r_b (mm)	k_{av} (W/m K)				
			#1	#2	#3	#4	#5
0.4	0.04	10.0	40.77	74.45	61.91	139.36	326.77
	0.05		46.50	79.00	63.67	135.00	315.83
	0.06		51.98	83.35	65.35	130.83	305.37
	0.07		57.22	87.51	66.96	126.83	295.35
	0.08		62.25	91.50	68.50	123.00	285.75
0.4	0.04	8.5	42.29	75.66	62.37	138.21	323.88
		6.5	45.40	78.13	63.33	135.84	317.94
		5.0	49.36	81.27	64.55	132.82	310.36
0.5	0.05	10.0	40.77	74.45	61.91	139.36	326.77
0.6	0.06	10.0	40.77	74.45	61.91	139.36	326.77

As for the thermal conditions, temperatures of the fin base and the fluid have been fixed to $T_b = 200$ °C and $T_\infty = 30$ °C, respectively. Of course, the problem is homogeneous, so that specific temperature values do not have any influence; a reasonable difference $T_b - T_\infty = 170$ K has the sole effect of assuring adequate levels of total heat transfer (W/m or W) in view of the precision of the numerical calculation. Convection coefficient has been given values within the reasonable interval $h = 40$–100 W/m^2 K.

5. Numerical method

Adopted case-studies have been solved numerically by the Finite Element Method (FEM). This linear, steady, 2D conduction problem is conveniently handled by the simplest form of FEM, encompassing triangular elements and linear interpolating functions, exactly as described in [23]. Also the FEM is more suited to curved boundaries and moderate disparity of length scales, as it is the case.

Computed runs comprised a grand total of 17 geometries times five material pairs, i.e., the 85 cases shown in Tables 2 and 3, times (typically) 5 values of the convection coefficient. In order to ease such an amount of computations, the commercial code FEHT has been used [24], which provides automatic grid definition and refinement, among other features. With a minimum value of the geometric ratio $t_c/t_s = 0.1$, proper attention has been paid when meshing the coating, in order to avoid too elongated elements. A grid of 3 840 triangular elements has been used in the calculation of half the fin geometry, providing 2000 nodes for temperature values. In all cases, one third of the elements are located in the coating and the rest in the substrate (the coating domain ranges from 11% to 21% of the total in the straight fin and from 13% to 24% in the annular fin). Calculation of heat dissipation was effected through piece-wise linear interpolation of the computed boundary temperatures.

To assure numerical accuracy in the results, grid refinement studies were undertaken. For this kind of problem, a simple criterion suffices; accordingly, mesh spacing was reduced by a factor of 4 for selected cases and the (dimensional) total heat transfer results compared. For the most unfavorable cases of the highest Biot numbers, discrepancies were always lower than 0.6%, thus confirming the grid-independence of the calculations.

Finally, for the 1D approximation, the parameter m_{av} is given by Eqs. (13) and (16), where thicknesses t and t_s must be understood at the base of the fin. Formulae that substitute Eq. (7) for fins of variable thickness are found in textbooks [17]. In the case of a straight fin of trapezoidal profile:

$$\eta = \frac{1}{mL}\frac{K_1(2mL_e^{\frac{1}{2}}L_t^{\frac{1}{2}})I_1(2mL_e) - I_1(2mL_e^{\frac{1}{2}}L_t^{\frac{1}{2}})K_1(2mL_e)}{K_1(2mL_e^{\frac{1}{2}}L_t^{\frac{1}{2}})I_0(2mL_e) + I_1(2mL_e^{\frac{1}{2}}L_t^{\frac{1}{2}})K_0(2mL_e)} \quad (18)$$

and for the annular hyperbolic fin:

$$\eta = \frac{2r_b}{m(r_t^2 - r_b^2)} \times \frac{I_{\frac{2}{3}}\left(\frac{2}{3}mr_b\right)I_{-\frac{2}{3}}\left(\frac{2}{3}mr_t[r_t/r_b]^{\frac{1}{2}}\right) - I_{-\frac{2}{3}}\left(\frac{2}{3}mr_b\right)I_{\frac{2}{3}}\left(\frac{2}{3}mr_t[r_t/r_b]^{\frac{1}{2}}\right)}{I_{\frac{1}{3}}\left(\frac{2}{3}mr_b\right)I_{\frac{2}{3}}\left(\frac{2}{3}mr_t[r_t/r_b]^{\frac{1}{2}}\right) - I_{-\frac{1}{3}}\left(\frac{2}{3}mr_b\right)I_{-\frac{2}{3}}\left(\frac{2}{3}mr_t[r_t/r_b]^{\frac{1}{2}}\right)} \quad (19)$$

In Eq. (18), we have used the fictitious length L_e for simplicity, see Fig. 1. The approximation of the composite fin consists in substituting m_{av} for m, as exemplified by Eq. (12c).

6. Results and discussion

We begin with some parametric studies under varying dimensional magnitudes. Figs. 3 and 4 represent fin efficiency for fixed substrate geometry and heat transfer conditions (and curvature for the annular fin), increasing coating thicknesses. In Figs. 5 and 6, a fixed geometry for the whole composite is examined under variable heat transfer coefficients. Fig. 7 shows an annular fin of constant length and thicknesses for variable curvature or base radius r_b and $m_{av}L = \text{const.}$ (which is achieved by adjusting the coefficient h).

Some trends in efficiency are those of an ordinary fin, decreasing with convection coefficient and increasing with radius. Additionally, in Figs. 3 and 4 we see the effect of increasing the thickness of the coating. If it is more conductive than the substrate (combinations #1, #2, #3 in Table 1), an increased efficiency results, at a rate that grows with the ratio k_c/k_s. This is quite pronounced for pair #1, whose ratio is the highest by an order of magnitude. In the opposite situation (combinations #4, #5), a slight decrease obtains. Concerning the relative performance of different material pairs, the order consistently seen in the figures is, from low to high: #1, #3, #2, #4, #5. Looking at Table 1, we note that this is dictated by the conductivity of the thicker substrate material. (For pairs #2 and #3 the coating conductivity seems to decide, the difference being the lowest.)

Figs. 3–7 show both the 2D numerical calculation and the 1D approximation. Agreement is good for the whole range of cases tested. If we define a percent error as

$$\varepsilon = 100\frac{\eta_a - \eta_n}{\eta_n} \quad (20)$$

maximum values are 1.75% and 1.99% for the straight and annular fins, respectively. The graphs show that errors diminish with efficiency, which would indeed make sense, indicating that the 1D approximation is more accurate

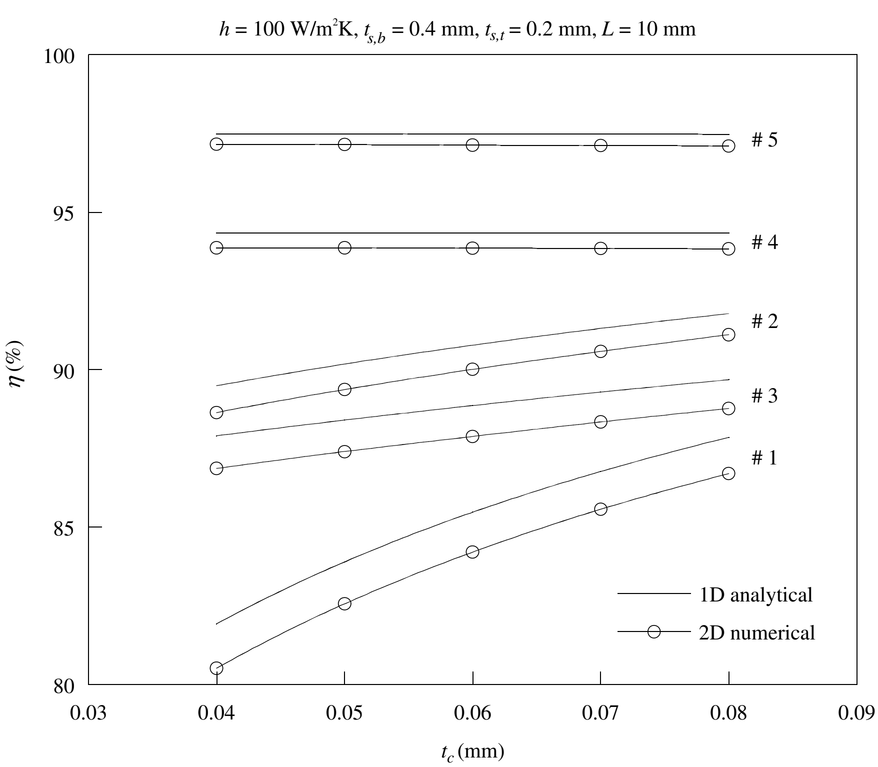

Fig. 3. Efficiency of the straight fin of trapezoidal profile for variable coating thickness.

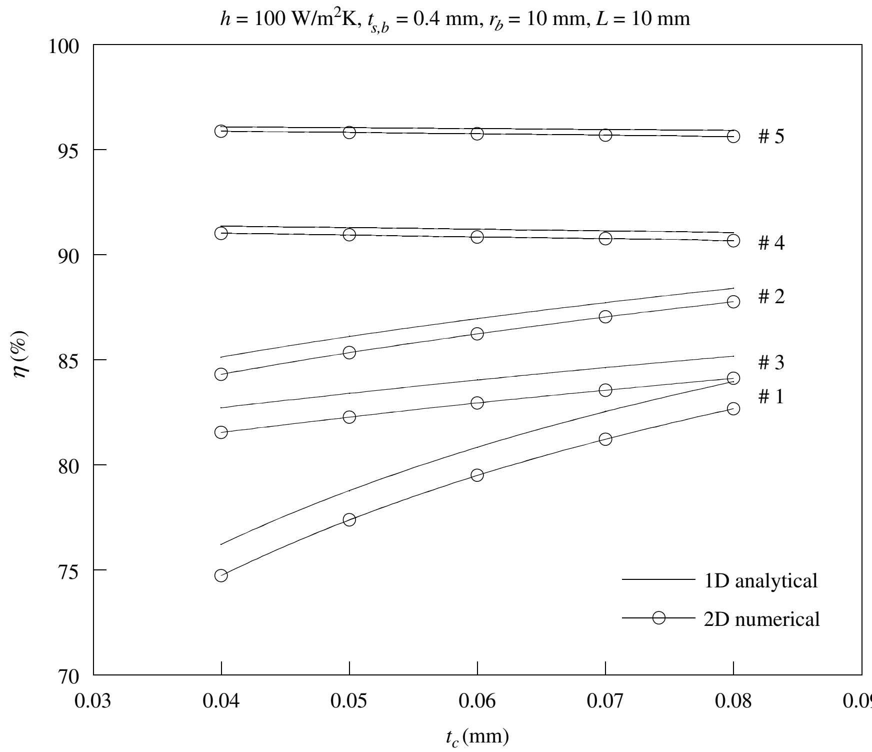

Fig. 4. Efficiency of the annular fin of hyperbolic profile for variable coating thickness.

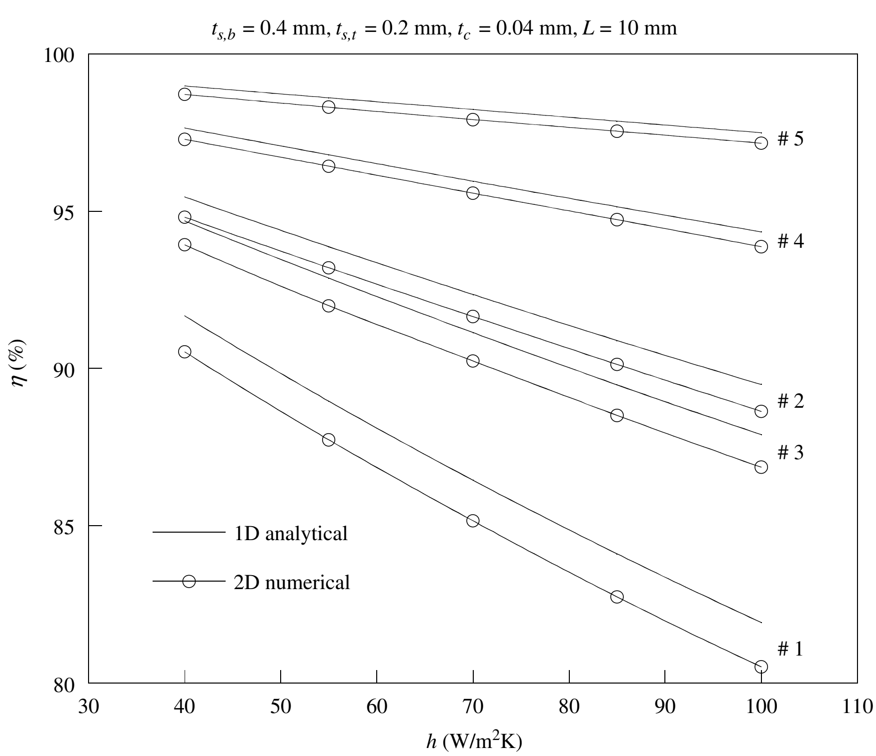

Fig. 5. Efficiency of the straight fin of trapezoidal profile for variable convection coefficient.

the lowest the temperature gradients in the composite material. In fact, maximum discrepancies arise for efficiencies lower than 80%, mostly an off-design figure for normal fin applications. Thus, accuracy of the approximation in practice will be even better than our maximum error figures suggest.

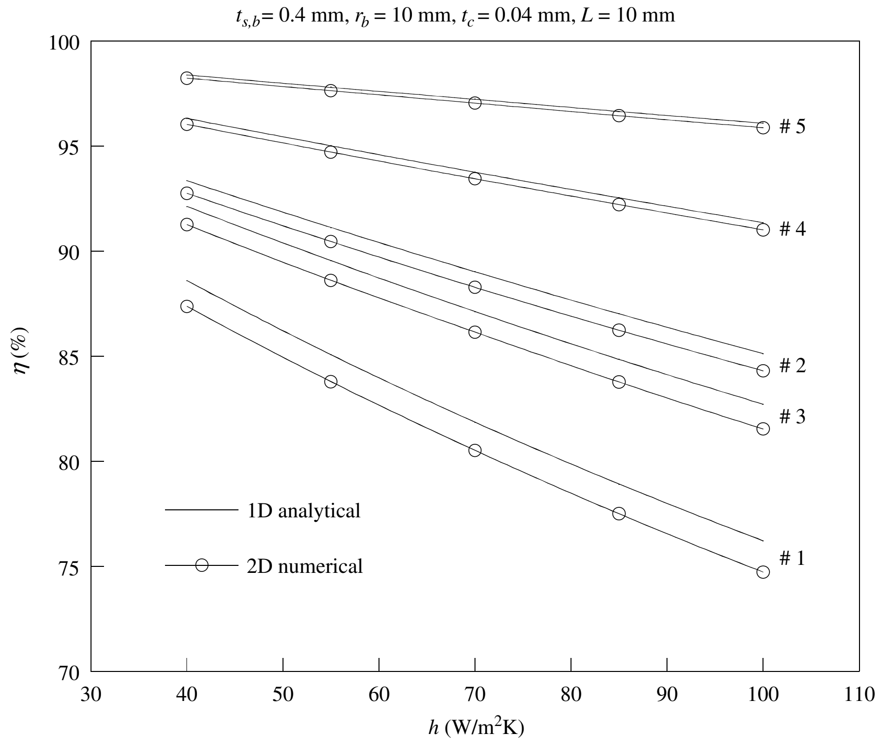

Fig. 6. Efficiency of the annular fin of hyperbolic profile for variable convection coefficient.

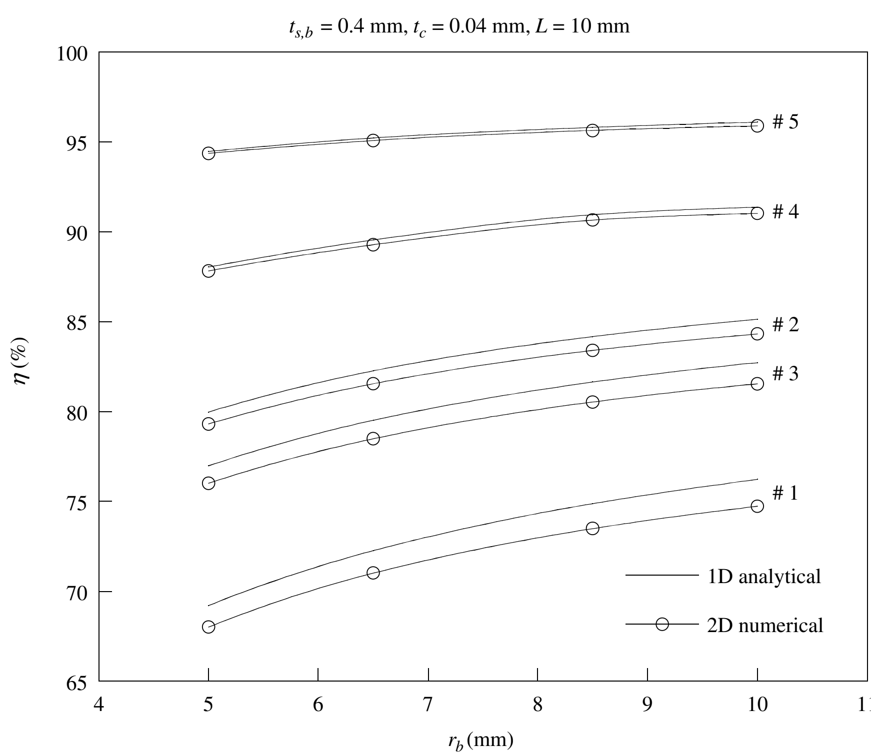

Fig. 7. Efficiency of the annular fin of hyperbolic profile for variable curvature.

On the other hand, since the magnitude of the error is strongly related to efficiency, and this in turn varies more with material combination than it does with geometrical and thermal parameters, it is not easy to ascertain its behavior. For example, larger errors arise for material pair #1, when the average thermal conductivity is the lowest, but also the ratio k_c/k_s is the largest. For the case-studies of Figs. 3–6, the magnitude of Bi_{av}, Eq. (17), ranges from 0.53×10^{-4} to 1.181×10^{-3}. Now, if the error ε is represented versus this parameter in Fig. 8, an appreciable scatter is present and some peculiar trends can be distinguished, but the conclusion is clear: The error grows quite reasonably as a one-to-one function of the Biot number.

Fig. 9 represents the same fin efficiencies, but now against the average thermal length $m_{av}L$ in each case. This shows more simply the 1D approximation: there is almost

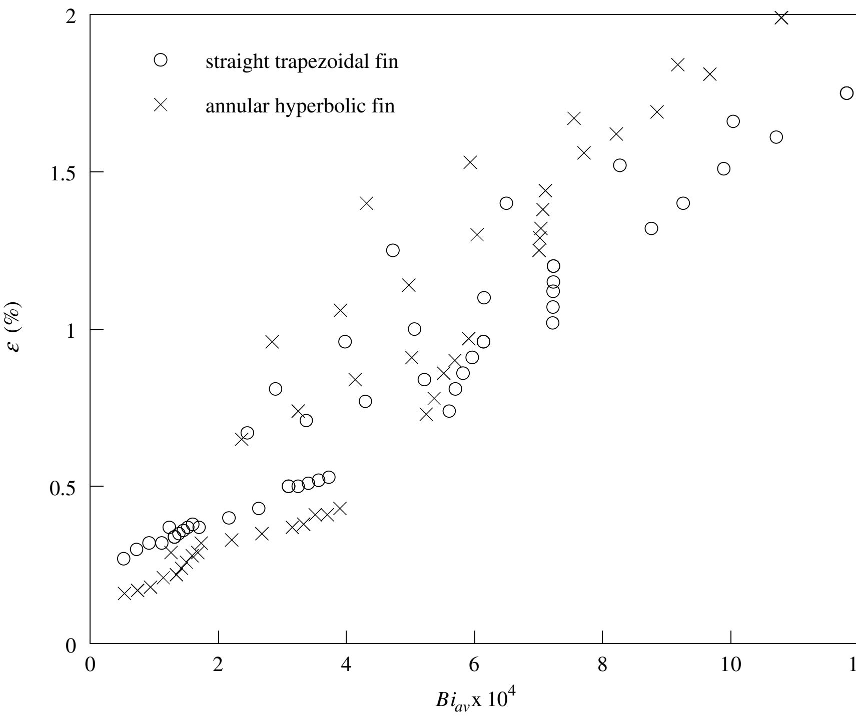

Fig. 8. Error ε versus average Biot number.

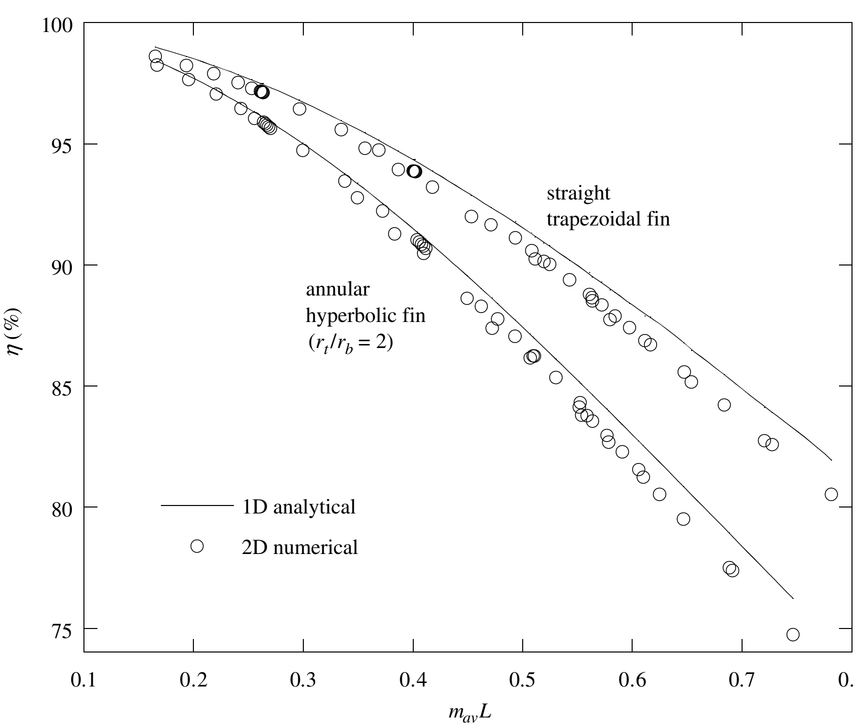

Fig. 9. Exact and approximate curves of fin efficiency versus thermal length.

Table 4
Influence of total slenderness $(t_{s,b}+t_c)/L$ and geometric ratio $t_c/t_{s,b}$ on 2D fin efficiency

Material	t_c (mm)	$\frac{t_c}{t_{s,b}}$	$\frac{t_{s,b}+t_c}{L}$	Straight trapezoidal fin		Annular hyperbolic fin	
				$m_{av}L$	η(%)	$m_{av}L$	η(%)
#1	0.04	0.10	0.044	0.7813	80.52	0.7466	74.73
	0.05		0.055		80.48		74.67
	0.06		0.066		80.45		74.62
#2	0.04	0.10	0.044	0.5632	88.64	0.5525	84.31
	0.05		0.055		88.59		84.24
	0.06		0.066		88.54		84.18
#5	0.04	0.10	0.044	0.2610	97.16	0.2637	95.88
	0.05		0.055		97.09		95.79
	0.06		0.066		97.04		95.70
#1	0.04	0.10	0.044	0.7813	80.52	0.7466	74.73
	0.06	0.15			80.60		74.82
	0.08	0.20			80.68		74.92
#2	0.04	0.10	0.044	0.5632	88.64	0.5525	84.31
	0.06	0.15			88.73		84.43
	0.08	0.20			88.82		84.56
#5	0.04	0.10	0.044	0.2610	97.16	0.2637	95.88
	0.06	0.15			97.04		95.72
	0.08	0.20			96.92		95.56

no scatter in the numerical, 2D values (dots), that seem to closely follow a single curve, also very close to the 1D formula (continuous line). Although some slight curvature is perceived in the graphs, discrepancies clearly increase with decreasing efficiencies, or, as seen in the figures, with increasing thermal lengths.

Another remarkable fact evident in Fig. 9 is that the 1D approximation, as given by the analytical fin formulae combined with Eqs. (13) and (16), seems to provide a consistent upper bound for the 2D values. This has been theoretically demonstrated for single-material fins [25]; here, it additionally suggests the adequacy of the arithmetical, volume-averaged thermal conductivity [7]. It can be seen indeed, for our annular geometry, that an average with the area of the profile, which is slightly different in this case, leads to equally satisfactory approximations but lacks this property.

Fig. 9 also suggests that some approximation in the generic form of Eq. (12b) can be better than Eq. (12c), or in other words, that the definitions of average thermal conductivity and length can be themselves optimized. To explore this possibly, in one of the many forms that it can be done, we attempt to evaluate the errors arising when passing from Eq. (11) to Eq. (12a), and then to Eq. (12b), to see whether these reduce substantially the inaccuracies found thus far.

In Table 4, the influences of total slenderness $(t_{s,b}+t_c)/L$ and geometric ratio $t_c/t_{s,b}$ on the 2D efficiency are elucidated separately. Firstly, the fin is made up to 50% thicker proportionally maintaining $t_c/t_{s,b}=0.1$ and thus $k_{av}=\text{const}$. A thermal length $m_{av}L=\text{const}$. is then imposed by adjusting the coefficient h. In the second place, the coating thickness t_c is doubled over the same substrate, which gives a ratio $t_c/t_{s,b}$ varying from 0.10 to 0.20. Since the coating is much thinner, the ratio $(t_{s,b}+t_c)/L$ is approximately constant, thus avoiding the need to include

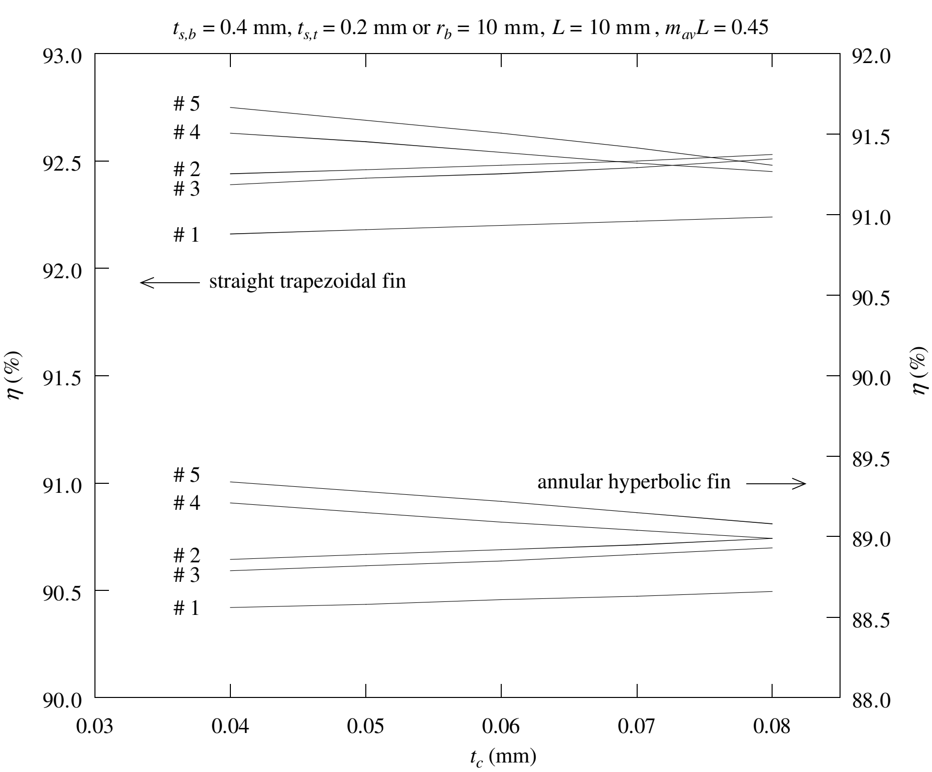

Fig. 10. Fin efficiency under constant thermal length.

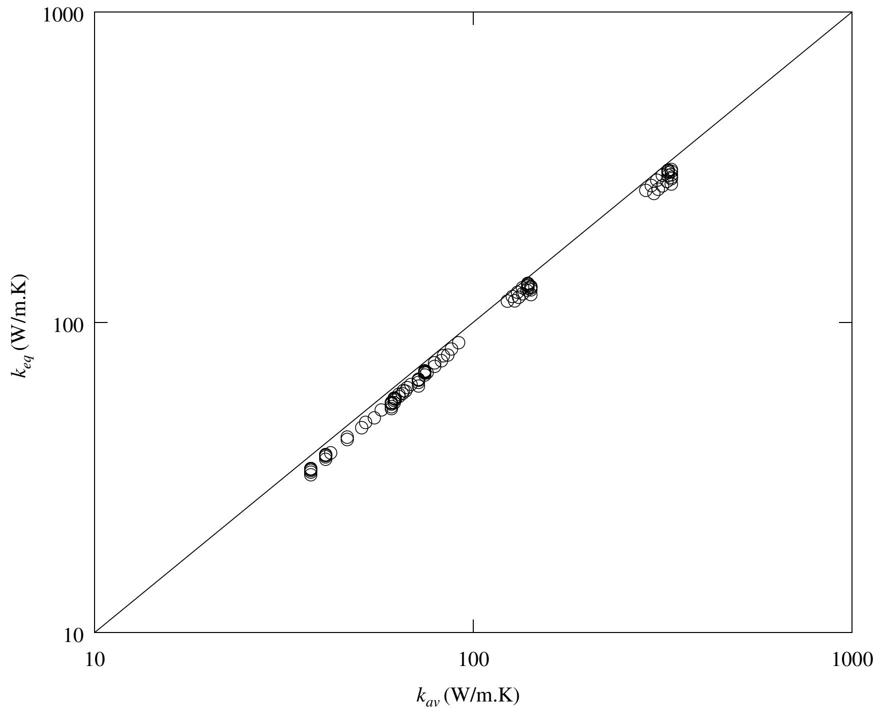

Fig. 11. Equivalent and volume-weighted averaged conductivities.

geometries with different L; the ensuing deviation in efficiency can be shown to be an order of magnitude lower than that induced by the variation of $t_c/t_{s,b}$. The coefficient h is again adjusted to give $m_{av}L = \text{const}$. Three different values of the parameter are specified in both cases, and calculations are repeated for the straight and annular geometries and for material pairs #1, #2, #5, which gives extreme and middle values of η.

As can be seen in Table 4, the three corresponding values of fin efficiency for each case do not differ greatly. Errors are in the range 0.07–0.12% for the first change and 0.16–0.24% for the second. Therefore, under reasonable ranges of variation, neglecting total slenderness and simplifying Eq. (11) to Eq. (12a) is completely reasonable, as in ordinary fins. Further neglecting the geometric ratio in Eq. (12a) roughly introduces a doubled uncertainty, but this is still of course a very low figure.

Finally, to isolate the influence of the conductivity ratio k_c/k_s, we simulate the 2D efficiency for all material pairs with a fixed substrate and variable coating thickness, adjusting one more time the convection coefficient to impose a fixed value of the thermal length. Results are shown in Fig. 10. Maximum deviations are on the side of the lowest $t_c/t_{s,b}$, amounting to 0.8% over the average. This indicates that k_c/k_s is the highest secondary influence in Eq. (12a), but indeed secondary: the simplification to Eq. (12b) is clearly warranted. On the other hand, this deviation approximately halves the figure found above for the error introduced by 1D formulae of the class of Eq. (12c).

Of course, all the previous advantages are rather scarce over a value of 2% maximum uncertainty. Consequently, although there are better approximations to the composite fin, the intuitive expedient we have examined in this paper is already of enough accuracy in practical terms. To further appreciate the approximation, we have determined an equivalent fin parameter $m_{eq}L$ by numerically solving the equation resulting from equating the fin 1D formula to the FEM-computed value of fin efficiency. Subsequently, an equivalent conductivity k_{eq} has been computed from the assumed relationship

$$m_{eq}L = \left[\frac{h}{k_{eq}(t_{s,b} + t_c)}\right]^{\frac{1}{2}} L \tag{21}$$

Fig. 11 is a graph of k_{eq} versus the conductivity k_{av} defined in Eq. (16), showing results corresponding to the cases of Fig. 9. As discussed above, the value $m_{eq}L$ is consistently higher than the corresponding $m_{av}L$, and thus k_{eq} is always lower than k_{av}. However, differences are very low, so that the average conductivity defined in Eq. (16) constitutes a good approximation.

7. Conclusions

Efficiency of composite fins of tapered profile can be accurately predicted by a single thermal length, as for simple fins. The most convenient approach is the ordinary 1D approximation, calculating the fin parameter with the total thickness and a volume-averaged thermal conductivity, Eqs. (13) and (16). This has been demonstrated for practical fin geometries, dimensions and composite material pairs, under the following conditions, also representative of common practice: maximum Biot number of 1.12×10^{-3}, thickness ratios from 0.1 to 0.2, conductivity ratios from 0.04 to 14, minimum efficiency of 80%. Maximum errors are always lower than 2%. This generalizes the method of calculation of composite fins; in fact, since the 1D approximation is predictably more accurate for constant thickness, our ranges and error figures can be applied to any composite fin, of constant or variable profile. Of course, it is understood that standard assumptions of fin theory apply; other effects, such as non-uniformity of base temperature, thermal conductivities or convection coefficient should be studied separately, as in ordinary fins.

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